

# Q-deformation, discrete time and quantum information as fiber space

Jaroslav Hruby

Institute of Physics AV CR, Czech Republic

*e-mail: hruby.jar@centrum.cz*

## Abstract

In this paper we show the connection between the q-deformation and discrete time, starting from the q-deformed Heisenberg uncertainty relation and q-deformation calculus. We show that time has discrete nature and for this case we construct the connection between quantum information and spacetime via fiber space structure.

## 1 Heisenberg uncertainty relation, q-deformation and discretization of spacetime

Following the papers [1] let observables  $Q$  and  $P$  fulfill:

$$\mathbf{a} = 1/\sqrt{2}(\mathbf{Q} + i\mathbf{P}) , \quad \mathbf{a}^+ = 1/\sqrt{2}(\mathbf{Q} - i\mathbf{P}) \quad (1)$$

where  $\mathbf{a}$  and  $\mathbf{a}^+$  are creation and annihilation operators as usual and  $[\mathbf{Q}, \mathbf{P}] = i$ .

We shall define  $\Delta P \equiv P - \langle P \rangle$  and  $\Delta Q \equiv Q - \langle Q \rangle$ . Then the famous Heisenberg uncertainty relation for observables  $Q$  and  $P$ :

$$\begin{aligned} \frac{1}{2} |\langle [\mathbf{Q}, \mathbf{P}] \rangle| &= \frac{1}{2} |\langle [\Delta \mathbf{Q}, \Delta \mathbf{P}] \rangle| \\ &\leq \langle \Delta \mathbf{Q}^2 \rangle^{\frac{1}{2}} \langle \Delta \mathbf{P}^2 \rangle^{\frac{1}{2}} . \end{aligned} \quad (2)$$

The uncertainty can be understood as the product  $\Delta Q \Delta P$  of standard deviations of two observables for a system in a given state and it is often used to identify it with noncommutativity of quantum observables under consideration.

If we shall assume the q-deformation of the commutator between creation and annihilation operator we can write :

$$\mathbf{a} \mathbf{a}^+ - q \mathbf{a}^+ \mathbf{a} = I , \quad (3)$$

where  $q$  is the deformation parameter  $0 < q \leq 1$  and  $\mathbf{I}$  is identity operator. Let  $\mathbf{P}$  and  $\mathbf{Q}$  are Hermitean operators, which via  $\mathbf{a}$  and  $\mathbf{a}^+$  have form:

$$\mathbf{P} = \alpha \mathbf{a} + \alpha^* \mathbf{a}^+ , \quad \mathbf{Q} = \beta \mathbf{a} + \beta^* \mathbf{a}^+ , \quad (4)$$

where  $\alpha, \beta$  are complex parameters.

Then from q-commutation relation (3) follows:

$$[\mathbf{P}, \mathbf{Q}] = (\alpha\beta^* - \alpha^*\beta)[\mathbf{I} + (q-1)\mathbf{a}^+\mathbf{a}] = \mathbf{R} . \quad (5)$$

We can see for  $q = 1$  and  $\alpha(\beta)^* - (\alpha)^*\beta = -i$  that (5) are ordinary uncertainty relation.

Uncertainty relation follows directly from (5)

$$\frac{1}{4} |\langle \mathbf{R} \rangle|^2 \leq \langle \Delta \mathbf{Q}^2 \rangle \langle \Delta \mathbf{P}^2 \rangle , \quad (6)$$

and it is the known form for operators fulfilling  $[\mathbf{Q}, \mathbf{P}] = i\mathbf{R}$ .

For operators  $\mathbf{Q}, \mathbf{P}, \mathbf{R}$  the q-deformed uncertainty relation are:

$$\langle \Delta \mathbf{Q}^2 \rangle = |\beta|^2 [1 + (q-1)|\langle \mathbf{a} \rangle|^2] , \quad (7)$$

$$\langle \Delta \mathbf{P}^2 \rangle = |\alpha|^2 [1 + (q-1)|\langle \mathbf{a} \rangle|^2] , \quad (8)$$

$$\langle \mathbf{PQ} \rangle - \langle \mathbf{QP} \rangle = (\alpha\beta^* - \beta^*\alpha)[1 + (q-1)|\langle \mathbf{a} \rangle|^2] . \quad (9)$$

It is important that (7)–(9) are valid for arbitrary operators which are fulfilling (6) and representing conjugate physical observables.

It was demonstrated that the more basic notions of expected value, variance and uncertainty relation also have a clear geometric interpretation. This interpretation is based directly on the association of observables with vector fields on the sphere of states and does not employ the Hamiltonian formalism on the phase space. This makes the interpretation particularly transparent

and naturally leads one to a geometric uncertainty identity. Here one is faced then a new point of view on quantum mechanics (QM) that makes that theory quite similar to Einstein's general relativity.

It is also well known that uncertainty relation in q-QM for coordinate  $x$  and  $p$  can be obtain via the way :

let Bargmann-Fock's operators have the form

$$a = \frac{1}{2L} x - \frac{i}{2K} p , \quad (10)$$

$$a^+ = \frac{1}{2L} x + \frac{i}{2K} p = \partial_a , \quad (11)$$

and  $q$ -commutator

$$a a^+ - q_0^2 a^+ a = [a, \partial_a]_{q_0} = 1 \quad (12)$$

wher  $q_0$  is real deformation parameter connected with the constants  $K$  and  $L$  via  $KL = \frac{\hbar}{4} (q_0^2 + 1)$  . Here the constants  $L$  and  $K$  have the dimension of length and impuls.

The commutation relation between  $x$  a  $p$  is

$$[x, p] = i \hbar (1 + f(q_0, x, p)) , \quad (13)$$

where

$$f(q_0, x, p) = \frac{q_0^2 - 1}{4} \left( \frac{x^2}{L^2} + \frac{p^2}{K^2} \right) . \quad (14)$$

Then the  $q$ -deformed Heisenberg uncertainty relation follows

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + f(q_0, (\Delta x)^2 + \langle x \rangle^2, (\Delta p)^2 + \langle p \rangle^2) \right] . \quad (15)$$

In every case of  $q$ -deformed QM we have minimal uncertainty in  $x$  and also in  $p$ , which are for  $q_0 > 1$  :

$$\Delta x_0 = L \sqrt{1 - q_0^{-2}} , \quad \Delta p_0 = K \sqrt{1 - q_0^{-2}} . \quad (16)$$

It gives the way to the discretization of the spacetime in  $q$ -deformed world.

## 2 Q-deformation, discrete time and quantum information

Let us suppose that  $q_E$  is the parameter of the discretization of spacetime.

Let us consider the discretization of standard differential calculus in one space dimension

$$[x, dx] = dxq_E, \quad (17)$$

and the action of the discrete translation group

$$x^n dx = dx(x + q_E)^n, \quad (18)$$

$$\psi(x)dx = dx\psi(x + q_E), \quad (19)$$

for any wave function  $\psi$  of the Hilbert space of QM with the discrete space variable.

The discrete space variable is defined as  $x = nq_E$ , where  $n$  is an integer and  $q_E$  is the interval between two discrete space points in this space variable.

If we define the derivatives by

$$d\psi(x) = dx(\partial_x \psi)(x) = (\overleftarrow{\partial} \psi)(x)dx, \quad (20)$$

$$(\partial_x \psi)(x) = \frac{1}{q_E}[\psi(x + q_E) - \psi(x)], \quad (21)$$

$$(\overleftarrow{\partial}_x \psi)(x) = \frac{1}{q_E}[\psi(x) - \psi(x - q_E)], \quad (22)$$

$$(\overleftarrow{\partial}_x \psi)(x) = (\partial_x \psi)(x - q_E), \quad (23)$$

then the ordinary one-dimensional Schrödinger equation will be

$$\frac{1}{2} \frac{d^2 \psi(x)}{dx^2} + [E - U(x)]\psi(x) = 0, \quad (24)$$

with the potential  $U(x)$  and wavefunction  $\psi(x) \equiv \psi(E, x)$ , corresponding to energy value  $E$ , has on the discrete space the form

$$\frac{1}{2l^2}[\psi((n+1)q_E) - 2\psi(nq_E) + \psi((n-1)q_E)] + [E - U(nq_E)]\psi(nq_E) = 0. \quad (25)$$

We now show the coincidence between such discretization model, non-commutative differential calculus and q-deformed QM, assuming  $q^2 \approx 1$ .

Let us suppose that ordinary continuum space variable  $y$  in QM has the form:

$$y = \lim_{q_E \rightarrow 0} (1 + q_E)^{\frac{x}{q_E}} = e^x. \quad (26)$$

Using Eqs.(2.4-2.7) and (2.10) we get:

$$\partial_y = y^{-1} \partial_x = (q_E + 1)^{\frac{-1}{q_E}} \partial_x \quad (27)$$

Thus, using  $q_E \equiv q^2 - 1$ , we have

$$(\partial_y \psi)(y) = \frac{\psi((q_E + 1)y) - \psi(y)}{q_E y} = \frac{\psi(q^2 y) - \psi(y)}{(q^2 - 1)y} \quad (28)$$

$$(\overleftarrow{\partial}_y \psi)(y) = (q_E + 1) \frac{\psi(y) - \psi((q_E + 1)y)}{q_E y} = \frac{\psi(y) - \psi(q^2 y)}{(1 - q^{-2})y} \quad (29)$$

what represents derivatives in the differential on the quantum hyperplane.

We can see that for  $q_E = 0$  or  $q^2 = 1$  we have the ordinary QM and continuous space-time.

### 3 Q-deformation calculus, non-commutativity and differential geometry

Aspects of gauge theory, Hamiltonian mechanics and QM arise naturally in the mathematics of a non-commutative framework for calculus and differential geometry.

Following summary paper [2] we can see that the q-deformation calculus has the deep connection with differential geometry and gauge fields.

There is shown the constructions of the non-commutativity are performed in a Lie algebra  $\mathcal{A}$ . One may take  $\mathcal{A}$  to be a specific matrix Lie algebra, or abstract Lie algebra. If  $\mathcal{A}$  is taken to be an abstract Lie algebra, then it is convenient to use the universal enveloping algebra so that the Lie product can be expressed as a commutator. In making general constructions of operators satisfying certain relations, it is understood that one can always begin with a free algebra and make a quotient algebra where the relations are satisfied.

On  $\mathcal{A}$ , a variant of calculus is built by defining derivations as commutators (or more generally as Lie products). For a fixed  $N$  in  $\mathcal{A}$  one defines

$$\nabla_N : \mathcal{A} \longrightarrow \mathcal{A}$$

by the formula

$$\nabla_N F = [F, N] = FN - NF.$$

$\nabla_N$  is a derivation satisfying the Leibniz rule.

$$\nabla_N(FG) = \nabla_N(F)G + F\nabla_N(G).$$

There are many motivations for replacing derivatives by commutators. If  $f(x)$  denotes (say) a function of a real variable  $x$ , and  $\tilde{f}(x) = f(x + h)$  for a fixed increment  $h$ , define the *discrete derivative*  $Df$  by the formula  $Df = (\tilde{f} - f)/h$ , and find that the Leibniz rule is not satisfied. One has the basic formula for the discrete derivative of a product:

$$D(fg) = D(f)g + \tilde{f}D(g).$$

Correct this deviation from the Leibniz rule by introducing a new non-commutative operator  $J$  with the property that

$$fJ = J\tilde{f}.$$

Define a new discrete derivative in an extended non-commutative algebra by the formula

$$\nabla(f) = JD(f).$$

It follows at once that

$$\nabla(fg) = JD(f)g + J\tilde{f}D(g) = JD(f)g + fJD(g) = \nabla(f)g + f\nabla(g).$$

Note that

$$\nabla(f) = (J\tilde{f} - Jf)/h = (fJ - Jf)/h = [f, J/h].$$

In the extended algebra, discrete derivatives are represented by commutators, and satisfy the Leibniz rule. One can regard discrete calculus as a subset of non-commutative calculus based on commutators.

In  $\mathcal{A}$  there are as many derivations as there are elements of the algebra, and these derivations behave quite wildly with respect to one another. If one takes the concept of *curvature* as the non-commutation of derivations, then  $\mathcal{A}$  is a highly curved world indeed. Within  $\mathcal{A}$  one can build a tame world of derivations that mimics the behaviour of flat coordinates in Euclidean space. The description of the structure of  $\mathcal{A}$  with respect to these flat coordinates contains many of the equations and patterns of mathematical physics.

The flat coordinates  $X_i$  satisfy the equations below with the  $P_j$  chosen to represent differentiation with respect to  $X_j$ :

$$[X_i, X_j] = 0$$

$$[P_i, P_j] = 0$$

$$[X_i, P_j] = \delta_{ij}.$$

Derivatives are represented by commutators.

$$\partial_i F = \partial F / \partial X_i = [F, P_i],$$

$$\hat{\partial}_i F = \partial F / \partial P_i = [X_i, F].$$

Temporal derivative is represented by commutation with a special (Hamiltonian) element  $H$  of the algebra:

$$dF/dt = [F, H].$$

(For quantum mechanics, take  $i\hbar dA/dt = [A, H]$ .) These non-commutative coordinates are the simplest flat set of coordinates for description of temporal phenomena in a non-commutative world. Note:

**Hamilton's Equations.**

$$dP_i/dt = [P_i, H] = -[H, P_i] = -\partial H / \partial X_i$$

$$dX_i/dt = [X_i, H] = \partial H / \partial P_i.$$

These are exactly Hamilton's equations of motion. The pattern of Hamilton's equations is built into the system.

**Discrete Measurement.** Consider a time series  $\{X, X', X'', \dots\}$  with commuting scalar values. Let

$$\dot{X} = \nabla X = JDX = J(X' - X)/\tau$$

where  $\tau$  is an elementary time step (If  $X$  denotes a times series value at time  $t$ , then  $X'$  denotes the value of the series at time  $t + \tau$ ). The shift operator  $J$  is defined by the equation  $XJ = JX'$  where this refers to any point in the time series so that  $X^{(n)}J = JX^{(n+1)}$  for any non-negative integer  $n$ . Moving  $J$  across a variable from left to right, corresponds to one tick of the clock. This discrete, non-commutative time derivative satisfies the Leibniz rule.

This derivative  $\nabla$  also fits a significant pattern of discrete observation. Consider the act of observing  $X$  at a given time and the act of observing (or obtaining)  $DX$  at a given time. Since  $X$  and  $X'$  are ingredients in computing  $(X' - X)/\tau$ , the numerical value associated with  $DX$ , it is necessary to let the clock tick once. Thus, if one first observe  $X$  and then obtains  $DX$ , the result is different (for the  $X$  measurement) if one first obtains  $DX$ , and then observes  $X$ . In the second case, one finds the value  $X'$  instead of the value  $X$ , due to the tick of the clock.

1. Let  $\dot{X}X$  denote the sequence: observe  $X$ , then obtain  $\dot{X}$ .
2. Let  $X\dot{X}$  denote the sequence: obtain  $\dot{X}$ , then observe  $X$ .

The commutator  $[X, \dot{X}]$  expresses the difference between these two orders of discrete measurement. In the simplest case, where the elements of the time series are commuting scalars, one has

$$[X, \dot{X}] = X\dot{X} - \dot{X}X = J(X' - X)^2/\tau.$$

Thus one can interpret the equation

$$[X, \dot{X}] = Jk$$

( $k$  a constant scalar) as

$$(X' - X)^2/\tau = k.$$

This means that the process is a walk with spatial step

$$\Delta = \pm\sqrt{k\tau}$$



where  $k$  is a constant. In other words, one has the equation

$$k = \Delta^2/\tau.$$

This is the diffusion constant for a Brownian walk. A walk with spatial step size  $\Delta$  and time step  $\tau$  will satisfy the commutator equation above exactly when the square of the spatial step divided by the time step remains constant. This shows that the diffusion constant of a Brownian process is a structural property of that process, independent of considerations of probability and continuum limits.

**Heisenberg/Schrödinger Equation.** Here is how the Heisenberg form of Schrödinger's equation fits in this context. Let the time shift operator be given by the equation  $J = (1 + H\Delta t/i\hbar)$ . Then the non-commutative version of the discrete time derivative is expressed by the commutator

$$\nabla\psi = [\psi, J/\Delta t],$$

and we calculate

$$\begin{aligned}\nabla\psi &= \psi[(1 + H\Delta t/i\hbar)/\Delta t] - [(1 + H\Delta t/i\hbar)/\Delta t]\psi = [\psi, H]/i\hbar, \\ i\hbar\nabla\psi &= [\psi, H].\end{aligned}$$

This is exactly the Heisenberg version of the Schrödinger equation.

**Dynamics and Gauge Theory.** One can take the general dynamical equation in the form

$$dX_i/dt = \mathcal{G}_i$$

where  $\{\mathcal{G}_1, \dots, \mathcal{G}_d\}$  is a collection of elements of  $\mathcal{A}$ . Write  $\mathcal{G}_i$  relative to the flat coordinates via  $\mathcal{G}_i = P_i - A_i$ . This is a definition of  $A_i$  and  $\partial F/\partial X_i = [F, P_i]$ . The formalism of gauge theory appears naturally. In particular, if

$$\nabla_i(F) = [F, \mathcal{G}_i],$$

then one has the curvature

$$[\nabla_i, \nabla_j]F = [R_{ij}, F]$$

and

$$R_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

This is the well-known formula for the curvature of a gauge connection. Aspects of geometry arise naturally in this context, including the Levi-Civita connection (which is seen as a consequence of the Jacobi identity in an appropriate non-commutative world).

One can consider the consequences of the commutator  $[X_i, \dot{X}_j] = g_{ij}$ , deriving that

$$\ddot{X}_r = G_r + F_{rs}\dot{X}^s + \Gamma_{rst}\dot{X}^s\dot{X}^t,$$

where  $G_r$  is the analogue of a scalar field,  $F_{rs}$  is the analogue of a gauge field and  $\Gamma_{rst}$  is the Levi-Civita connection associated with  $g_{ij}$ . This decomposition of the acceleration is uniquely determined by the given framework.

## 4 Spacetime and information like a fiber space

We present a toy model where in every point of time exist an information, connected with the spacetime variation.

We shall call  $E$  with elements  $z^A$  in  $E$ , which are:

$$(x^\mu, \theta^\alpha) ; . \quad (30)$$

The fiber space is  $E(V, W, SU(2))$ , where the basis  $V$  is spacetime a fiber  $W$  is a information qubit.  $SU(2)$  is a Lie group acting on the fiber and  $E$  is the cartesian product of  $V$  and  $W$ .

On  $E$  we define one forms as usually:

$$\omega^\mu = \Omega^\mu + i\bar{\omega}^\alpha(\gamma^\mu)_{\alpha\beta}\theta^\beta , \quad (31)$$

$$\omega^\alpha = d\theta^\alpha . \quad (32)$$

For one discrete time dimension we have  $\theta^\alpha$  discrete time series.

For  $\Omega^\mu = dx^\mu$  is valid that  $\Omega_\mu\Omega^\mu$  is  $SU(2)$  invariant.

From fiber structure is known that  $\Omega^\mu$  is connected with arbitrary form  $\omega^\mu$  on  $E$  via following way:

$$\omega^\mu = \Omega^\mu + \bar{\omega}^\alpha\Gamma_\alpha^\mu , \quad (33)$$

where  $\Gamma_\alpha^\mu$  are connection forms on  $E$  and  $\Gamma$  are Pauli matrices.

This forms define transformation from one fiber to another when infinitesimal changes in the base are realized.

On in  $W$  is one-form  $\Omega^\mu$  and on  $V$  form  $\omega^\alpha$ . From  $x_\mu = \bar{\theta}^\mu\gamma^\mu\theta$  follows  $x^\mu$  is  $\theta^\alpha$  and  $\Omega^\mu$  has the form:

$$\Omega^\mu = dx^\mu$$

$$\begin{aligned}
&= \left( \frac{\partial x^\mu}{\partial \theta^\alpha} \right) d\theta^\alpha \\
&= \left( \frac{\partial x^\mu}{\partial \theta^\alpha} \right) \omega^\alpha .
\end{aligned}$$

It means that  $\Omega^\mu$  on  $W$  is given via  $\omega^\alpha$  on  $V$ .

It is valid:  $\frac{\partial x^\mu}{\partial \theta^\alpha} = \bar{\theta}'^\beta (\gamma^\mu)_{\beta\alpha}$ ,

and we get:  $\Omega^\mu = \bar{\theta}'^\beta (\gamma^\mu)_{\beta\alpha} \omega^\alpha$ .

Arbitrary object  $Y^J$  in  $E$  transforms as:

$$dY^J + Y_\mu^J \omega^\mu = \bar{\omega}^\alpha Y_\alpha^J . \quad (34)$$

Covariant derivation follows as

$$\nabla \Phi(x, \theta) = d\Phi(x, \theta) + \Phi_\mu(x, \theta) \omega^\mu + \Omega^\mu . \quad (35)$$

So we get:

$$\Omega^\mu = \omega^\mu - \bar{\omega}^\alpha \Gamma_\alpha^\mu$$

and then:

$$\nabla \Phi(x, \theta) = d\Phi(x, \theta) + \Phi_\mu(x, \theta) (\omega^\mu - \bar{\omega}^\alpha \Gamma_\alpha^\mu) . \quad (36)$$

We get:

$$\Phi_\mu(x, \theta) \omega^\mu = \bar{\omega}^\alpha \Phi_\alpha(x, \theta) - d\Phi(x, \theta)$$

and following:

$$\begin{aligned}
\nabla \Phi(x, \theta) &= \bar{\omega}^\alpha \Phi_\alpha(x, \theta) - \Phi_\mu(x, \theta) \bar{\omega}^\alpha \Gamma_\alpha^\mu \\
&= \bar{\omega}^\alpha (\Phi_\alpha(x, \theta) - \Phi_\mu(x, \theta) \Gamma_\alpha^\mu) \\
&= \bar{\omega}^\alpha \Phi_{i\alpha}(x, \theta) .
\end{aligned} \quad (37)$$

It is valid:

$$\Phi_{i\alpha}(x, \theta) = \partial_\alpha \Phi(x, \theta) - i(\gamma^\mu \theta)_\alpha \partial_\mu \Phi(x, \theta) = D_\alpha \Phi(x, \theta) ,$$

where  $D_\alpha = \partial_\alpha - i(\gamma^\mu \theta)_\alpha \partial_\mu = \partial_\alpha - \Gamma_\alpha^\mu \partial_\mu$  is the covariant derivation.

## 5 Conclusions

Here we show another aspect of the connection of the quantum time and quantum information. We show discrete nature of quantum time and present the idea of the nontrivial connection of quantum information and spacetime via fiber space.

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## References

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